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Multipliers on $H(p, q, \alpha)$ spaces

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Abstract Let f be an analytic function on the unit disc $\mathbb{D}$ and $F(a, b; c; z)$ be the Gaussian hypergeometric function. We consider the operator $T_{a,b,c}$ on $H(p, q, \alpha)$ defined as $T_{a,b,c} f(z) = f(z) * F(a, b; c; z)$, where $*$ denotes the usual Hadamard/convolution product. We prove that the Taylor coefficients of $F(a, b; c; z)$ are a multiplier from $H(p, q, \alpha)$ to $H(p, q, \alpha + a + b - c - 1)$ under certain conditions on a, b and c. As a consequence, we generalize some well-known results on fractional derivatives and integrals. Furthermore, we supply some conditions on a, b and c under which $F(a, b; c; z)$ lies in $H(p, q, \alpha)$.

الملخص

لتكن f دالة تحليلية على قرص الوحدة $\mathbb{D}$، وَ $F(a,b;c;z)$ دالة جاوس فوق هندسية. نعتبر المؤثر $T_{a,b,c}$ على $H(p,q,\alpha)$ المعرف كما يلي: $T_{a,b,c}f(z) = f(z) * F(a,b;c;z)$، حيث $*$ ضرب هادامارد الملفوف الاعتيادي. نثبت أن معاملات تايلور لِـ $F(a,b;c;z)$ هي مضروبة من $H(p,q,\alpha)$ إلى $H(p,q,\alpha+a+b-c-1)$ في ظل شروط معينة على a وَ b وَ c. نتيجة لذلك، نُعَمِّمُ بعض النتائج المعروفة جيداً على المشتقات والتكاملات الكسريَّة. بالإضافة إلى ذلك، نعطي بعض الشروط على a وَ b وَ c التي في ظلها يقع $F(a,b;c;z)$ في $H(p,q,\alpha)$.

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1 Introduction

Let f be analytic function on the unit disc $\mathbb{D}$. For $0 < p < \infty$, define for $r < 1$

$$M_p(f, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \mathrm{d}\theta \right)^{\frac{1}{p}}$$

and

$$M_\infty(f, r) = \max_{0 \le \theta \le 2\pi} |f(re^{i\theta})|.$$

Let $0 < p \le \infty$ and $0 < \alpha, q < \infty$, by $H(p, q, \alpha)$ we denote the space of analytic functions $f(z)$ on $\mathbb{D}$ satisfying

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$$\|f\|_{p,q,\alpha} = \left(\int_0^1 (1-r)^{q\alpha-1} M_p^q(f,r) \mathrm{d}r \right)^{\frac{1}{q}} < \infty.$$

For $0 < p \leq \infty$, $0 < \alpha < \infty$ and $q = \infty$, let H_α^p (or $H(p, \infty, \alpha)$) be the space of analytic functions $f(z)$ on $\mathbb{D}$ satisfying

$$\|f\|_{p,\infty,\alpha} = \sup_{0<r<1} (1-r)^\alpha M_p(f,r) < \infty.$$

Notice that $H\left(p, p, \frac{1}{p}\right) = A^p$ is the Bergman space, $H(p, \infty, 0) = H^p$ is the Hardy space, $H\left(1, 1, \frac{1}{p} - 1\right) = B^p$ is the Besov space and $H\left(p, q, \frac{1}{q-1}\right) = A^{p,q,\frac{1}{q-1}}$ is the weighted Bergman space.

Let X and Y be two spaces of analytic functions on the unit disc $\mathbb{D}$. Let $f \in X$ with $f(z) = \sum_{n=0}^\infty a_n z^n$. Suppose that $\{\lambda_n\}_{n=0}^\infty$ is a sequence such that $\sum_{n=0}^\infty \lambda_n a_n z^n \in Y$. Then we can define the operator $T_\lambda : X \to Y$ as $(T_\lambda f)(z) = \sum_{n=0}^\infty \lambda_n a_n z^n$. Then the sequence $\{\lambda_n\}_{n=0}^\infty$ is said to be a coefficient multiplier or simply multiplier from X into Y. The set of multipliers from X to Y is denoted by (X, Y).

It is an important question in function theory to describe the coefficient multipliers between various spaces of analytic functions in the unit disc $\mathbb{D}$. Coefficient multiplier is the way of obtaining information on the Taylor coefficients of functions in certain spaces making it possible to examine whether a given function is in a particular space. Multipliers between sequence spaces given by Taylor coefficients of analytic functions in $\mathbb{D}$ have been deeply studied in the literature. Since the time of Hardy and Littlewood, mixed norm and related spaces have been used to study function spaces on $\mathbb{D}$ and later to study multipliers between such spaces. Special emphasis has been put on the case where the spaces X and Y correspond to the sequence space of Taylor coefficient of analytic functions such as Hardy spaces, Bergman spaces, mixed norm spaces of analytic functions and so forth. The theory of the spaces $H(p, q, \alpha)$ was originated due to the work of Hardy and Littlewood (see [11,12]). Their work on $H(p, q, \alpha)$ spaces was continued by Flett and Sledd (see [9,23]) and later on by Pavlović (see [19,20]). Multipliers on Hardy spaces were in fashion for a long time and much work was done on them and related spaces. However nowadays complete descriptions of multipliers between Hardy spaces (H^p, H^q) for certain values of p and q remain still open. The reader is referred to the surveys (see [4,18]) for lots of results and references. Many results on multipliers of $H(p, q, \alpha)$, H^p, A^p, B^p and $A^{p,q,\alpha}$ have been established in the last decades (see [3,5,7,8,14–17,25,26] and thereby). For this purpose of study, the use of Gaussian hypergeometric function is an important tool.

The Classical/Gaussian hypergeometric function is defined by the power series expansion

$${}_2F_1(a, b; c; z) = \sum_{n=0}^\infty \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!} \qquad (|z| < 1).$$

Here a, b, c are complex numbers such that $c \neq -m$, $m = 0, 1, 2, 3, \ldots$ and (a, n) is the Pochhammer's symbol/shifted factorial defined by Appel's symbol

$$(a, n) := a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad n \in \mathbb{N}$$

and $(a, 0) = 1$ for $a \neq 0$ and Γ is the gamma function given by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t$. Obviously, $F(a, b; c; z)$ is analytic on the unit disc $\mathbb{D}$. Many properties of the hypergeometric series including the Gauss and Euler transformations are found in standard textbooks such as [1,24].

For any two analytic functions f and g represented by their power series expansions,

$$f(z) = \sum_{n=0}^\infty a_n z^n, \quad g(z) = \sum_{n=0}^\infty b_n z^n$$

in $|z| \leq R$, let $f * g$ denote the Hadamard product (or convolution) of f and g defined by

$$(f * g)(z) = \sum_{n=0}^\infty a_n b_n z^n \qquad \left(|z| < R^2\right).$$

Equivalently,

$$(f * g)(rz) = \frac{1}{2\pi} \int_0^{2\pi} f\left(ze^{-i\theta}\right) g\left(re^{i\theta}\right) \mathrm{d}\theta, \qquad 0 < r < 1.$$

Let f be an analytic function on unit disc $\mathbb{D}$. For $a, b, c \in \mathbb{C}$, we define

$$T_{a,b,c} f(z) = f(z) * F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} a_n z^n.$$

$T_{a,b,c}$ has a nice integral representation [2], which is given as follows

$$T_{a,b,c} f(z) = \int_0^1 \lambda(s) f(sz) \mathrm{d}s,$$

where

$$\lambda(s) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} s^{b-1}(1-s)^{c-a-b} F(c-a, 1-a; c-a-b+1; 1-s)$$

provided $Re\ a > 0$, $Re\ b > 0$, $Re\ (c+1) > Re\ (a+b)$. The operator $T_{a,b,c}(f)$ contains several well-known integral operators as its special cases [21]. For $c > b \geq 1$,

$$T_{1,b,c} f(z) \equiv T_{b,c} f(z) = \frac{1}{B(b, c-b)} \int_0^1 s^{b-1}(1-s)^{c-b-1} f(sz) \mathrm{d}s. \tag{1}$$

For instance, if $a = 1, b = 1+\beta, c = 1$, then $T_{1+\beta,1} = f^{[\beta]}$ and if $a = 1, b = 1, c = \beta+1$, then $T_{1,\beta+1} = f_{[\beta]}$ where the fractional derivative $f^{[\beta]}$ and fractional integral $f_{[\beta]}$ of order $\beta > 0$ are defined respectively by

$$f^{[\beta]}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\beta)}{\Gamma(n+1)} a_n z^n$$

and

$$f_{[\beta]}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\beta)} a_n z^n.$$

In terms of convolution, the above-mentioned definitions can be rewritten as

$$f^{[\beta]}(z) = \Gamma(1+\beta)[f(z) * F(1, 1+\beta; 1; z)]$$

and

$$f_{[\beta]}(z) = \frac{1}{\Gamma(1+\beta)}[f(z) * F(1, 1; 1+\beta; z)].$$

For more details on fractional derivative and integral (see [8–11]). Let us recall the result on fractional derivative [3, Theorem A].

Theorem: *Let $1 \leq p < \infty$, $0 < q, \alpha, \beta < \infty$. Then f belongs to $H(p, q, \alpha)$ if and only if $f^{[\beta]}$ belongs to $H(p, q, \alpha + \beta)$.*

The following result was proved by Duren, Shield et al. [8, Theorem 5].

Theorem: *Suppose $0 < p < q < 1$, and $\beta = \frac{1}{p} - \frac{1}{q}$.*

(a) If $f \in B^p$, then $f_{[\beta]} \in B^q$.
(b) If $f \in B^q$, then $f^{[\beta]} \in B^p$.

Recently in [13], Jevtić describes the multipliers spaces from $H(p, q, \alpha)$ to the space H^∞ of bounded analytic functions in the unit disc $\mathbb{D}$ and extends some results from [19] and [14]. In this paper, we are interested to define the operator $T_{a,b,c}$ and we find few conditions on a, b and c which guarantee that the Taylor coefficients of the Gaussian hypergeometric function $F\left(a, b; c; re^{i\theta}\right)$ form a multiplier from the spaces $H(p, q, \alpha)$ into the spaces $H(p, q, \alpha+\gamma)$ and $H(p, q, \alpha-\gamma)$. The main results form the content of Theorem 2.2, Lemma 2.4 and Theorem 2.5. The main techniques used in the proof of the results are Minkowski's inequality to switch the order of integration, monotonicity of integral mean and some known results and identities like those from Lemma 2.1 and Gauss identity [1]. The results are new and interesting as it unifies and gives new information about the behavior of functions in the mixed norm spaces under the action of the multiplier $T_{a,b,c}$. The results will extend those for fractional derivatives and integral due to the particular selection of the parameters. As corollaries of these results we obtain some known results by Blasco [3] and Duren et al. [8].

Throughout this paper C will denote a constant independent of f, depending only on indices p, q, α, etc., which may differ for different occurrences.

2 Main results

In this section we obtained conditions on a, b, c such that the operator $T_{a,b,c} f(z)$ defines a multiplier from $H(p, q, \alpha)$ spaces into $H(p, q, \alpha+\gamma)$ spaces. Also we obtained a result which is generalization of Theorem A of [3].

To prove our results, we recall some known results in the form of following lemmas.

Lemma 2.1 (i) *Let* $1 \le k < \infty, \mu > 0, \delta > 0, h : (0, 1) \to [0, \infty)$ *be measurable. Then*

$$\int_0^1 (1-r)^{k\mu-1} \left\{ \int_0^r (r-t)^{\delta-1} h(t)\mathrm{d}t \right\}^k \mathrm{d}r \le C \int_0^1 (1-r)^{k\mu+k\delta-1} h^k(r)\mathrm{d}r.$$

(ii) *For each* $p > 1$ *and* $z = re^{i\theta}$,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1-z|^p} \mathrm{d}\theta \le C(1-r)^{-(p-1)}.$$

(iii) *For* $0 < p < \infty$,

$$\int_0^{2\pi} \sup_{0 \le r < 1} |f(re^{i\theta})|^p \mathrm{d}\theta \le B\|f\|_p^p$$

with $B = B_p$ *independent of* $f \in H^p$.

(iv) *Let* $0 < p \le \infty, 0 < t < q < \infty$ *and* $\alpha > 0$. *Then*

$$\left(\int_0^1 (1-s)^{q\alpha-1} M_p^q(f, rs)\mathrm{d}s \right)^{\frac{1}{q}} \le C \left(\int_0^1 (1-s)^{t\alpha-1} M_p^t(f, rs)\mathrm{d}s \right)^{\frac{1}{t}}.$$

Lemma 2.1(ii) and (iii) are due to Hardy and Littlewood and Lemma 2.1(iii) is known as Hardy–Littlewood maximal theorem. For Lemma 2.1(ii) and (iii) we refer to pp. 65, 12 of [6]. For Lemma 2.1(i) we refer to pp. 758 of [9] and for Lemma 2.1(iv) we refer to Lemma 5 of [22].

Theorem 2.2 *Let* $1 \le p < \infty, 0 < q, \alpha, \gamma < \infty$. *Suppose* $a, b, c \in \mathbb{R}$ *with* $a+b > c+1$ *and* $\gamma = a+b-c-1$. *If* $f \in H(p, q, \alpha)$, *then* $T_{a,b,c} f(z) \in H(p, q, \alpha+\gamma)$.

Proof Using the definition of convolution we get

$$M_p\left(T_{a,b,c}f, r^2\right) = \left[\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{2\pi}\int_0^{2\pi} F\left(a,b;c;re^{it}\right) f\left(ze^{-it}\right)\mathrm{d}t\right|^p \mathrm{d}\theta\right]^{\frac{1}{p}},$$

where $z = re^{i\theta}$. Since $1 \le p < \infty$, Minkowski's inequality and in view of well-known Gauss identity [1], $F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z)$ we can rewrite the previous equation to obtain the following

$$\begin{aligned} M_p\left(T_{a,b,c}f, r^2\right) &\le \frac{1}{4\pi^2}\int_0^{2\pi}\left[\int_0^{2\pi}\left(\left|F\left(a,b;c;re^{it}\right)\right|\left|f\left(re^{i(\theta-t)}\right)\right|\right)^p \mathrm{d}\theta\right]^{\frac{1}{p}}\mathrm{d}t \\ &\le C M_p(f,r)\int_0^{2\pi}|1-re^{it}|^{c-a-b}\left|F\left(c-a,c-b;c;re^{it}\right)\right|\mathrm{d}t. \end{aligned}$$

Since $a+b>c$, $F(c-a,c-b;c;z)$ is bounded on $\mathbb{D}$ and so we have

$$M_p\left(T_{a,b,c}f, r^2\right) \le C M_p(f,r)\int_0^{2\pi}|1-re^{it}|^{c-a-b}\mathrm{d}t.$$

Therefore Lemma 2.1(ii) gives

$$M_p\left(T_{a,b,c}f, r^2\right) \le C M_p(f,r)(1-r)^{-(a+b-c-1)}.$$

which completes the proof. □

Theorem 2.3 *For $1 \le p < \infty$, $0 < q, \alpha, \gamma < \infty$, suppose $b, c \in \mathbb{R}$ with $b > c \ge 1$ and $\gamma = b - c$. Then $f \in H(p,q,\alpha)$ if and only if $T_{b,c}f \in H(p,q,\alpha+\gamma)$.*

Proof Let $f \in H(p,q,\alpha)$. $T_{b,c}f$ is equal to $T_{a,b,c}f$ when $a = 1$. So the proof for this part is the same as that of the previous theorem.

Conversely, suppose that $T_{b,c}f \in H(p,q,\alpha+\gamma)$. Here

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\frac{(b,n)(c,n)}{(c,n)(b,n)}a_n z^n \\ &= \frac{\Gamma(b)}{\Gamma(c)\Gamma(b-c)}\sum_{n=0}^{\infty}\frac{(b,n)}{(c,n)}a_n z^n\int_0^1 s^{c+n-1}(1-s)^{b-c-1}\mathrm{d}s \\ &= C\int_0^1 s^{c-1}(1-s)^{b-c-1}T_{b,c}f(sz)\mathrm{d}s. \end{aligned}$$

Therefore,

$$\begin{aligned} M_p(f,r) &\le C\int_0^1 s^{c-1}(1-s)^{b-c-1}M_p(T_{b,c}f, rs)\mathrm{d}s \\ &\le C\int_0^1 (1-s)^{\gamma-1}M_p(T_{b,c}f, rs)\mathrm{d}s. \end{aligned} \tag{2}$$

Case I. Let $0 < q \leq 1,\ q\gamma < 1$. Taking $t = q, q = 1$ in Lemma 2.1(iv), we have

$$M_p(f, r) \leq C \left(\int_0^1 (1-s)^{q\gamma-1} M_p^q(T_{b,c} f, rs) \mathrm{d}s \right)^{1/q}$$

$$r^{q\gamma} M_p^q(f, r) \leq C \int_0^r (r-s)^{q\gamma-1} M_p^q(T_{b,c} f, s) \mathrm{d}s$$

$$r M_p^q(f, r) \leq C \int_0^r (r-s)^{q\gamma-1} M_p^q(T_{b,c} f, s) \mathrm{d}s.$$

Hence,

$$\int_0^1 (1-r^2)^{\alpha q-1} r M_p^q(f, r) \mathrm{d}r \leq C \int_0^1 (1-r^2)^{\alpha q-1} \left(\int_0^r (r-s)^{q\gamma-1} M_p^q(T_{b,c} f, s) \mathrm{d}s \right) \mathrm{d}r.$$

Taking $k = 1,\ \mu = q\alpha,\ \delta = q\gamma$ and $h(s) = M_p^q(T_{b,c} f, s)$ in Lemma 2.1(i) completes the proof.

Case II. Let $0 < q \leq 1,\ q\gamma \geq 1$.

$$M_p^q(f, r) = C \frac{1}{r} \int_0^r \left(1 - \frac{s}{r}\right)^{q\gamma-1} M_p^q(T_{b,c} f, s) \mathrm{d}s$$

$$r M_p^q(f, r) \leq C \int_0^r (1-s)^{q\gamma-1} M_p^q(T_{b,c} f, s) \mathrm{d}s.$$

Hence,

$$\int_0^1 (1-r^2)^{\alpha q-1} r M_p^q(f, r) \mathrm{d}r \leq C \int_0^1 (1-r^2)^{\alpha q-1} \left(\int_0^r (1-s)^{q\gamma-1} M_p^q(T_{b,c} f, s) \mathrm{d}s \right) \mathrm{d}r.$$

Taking $k = 1,\ \mu = q\alpha,\ \delta = 1$ and $h(s) = (1-s)^{q\gamma-1} M_p^q(T_{b,c} f, s)$ in Lemma 2.1(i) gives the required inequality.

Case III. Let $1 < q < \infty$. By inequality (2) we have

$$\begin{aligned} \|f\|_{p,q,\alpha}^q &\leq C \int_0^1 (1-r)^{\alpha q-1} \left(\int_0^1 (1-s)^{\gamma-1} M_p(T_{b,c} f, rs) \mathrm{d}s \right)^q \mathrm{d}r \\ &= C \int_0^1 (1-r)^{\alpha q-1} \left(\int_0^r (1-s)^{\gamma-1} M_p(T_{b,c} f, rs) \mathrm{d}s + \int_r^1 (1-s)^{\gamma-1} M_p(T_{b,c} f, rs) \mathrm{d}s \right)^q \mathrm{d}r \\ &\leq I_1 + I_2, \end{aligned} \tag{3}$$

where,

$$I_1 = C 2^{q-1} \int_0^1 (1-r)^{\alpha q-1} \left(\int_0^r (1-s)^{\gamma-1} M_p(T_{b,c} f, rs) \mathrm{d}s \right)^q \mathrm{d}r$$

and

$$I_2 = C2^{q-1} \int_0^1 (1-r)^{\alpha q-1} \left(\int_r^1 (1-s)^{\gamma-1} M_p(T_{b,c} f, rs) \mathrm{d}s \right)^q \mathrm{d}r.$$

Therefore,

$$I_1 \leq C2^{q-1} \int_0^1 (1-r)^{\alpha q-1} \left(\int_0^r (1-s)^{\gamma-1} M_p(T_{b,c} f, s) \mathrm{d}s \right)^q \mathrm{d}r.$$

Taking $k = q,\ \mu = \alpha,\ \delta = 1$ and $h(s) = (1-s)^{\gamma-1} M_p(T_{b,c} f, s)$ in Lemma 2.1(i) gives

$$I_1 \leq C2^{q-1} \int_0^1 (1-r)^{q(\alpha+\gamma)-1} M_p^q(T_{b,c} f, r) \mathrm{d}r < \infty.$$

Since integral mean $M_p^q(T_{b,c} f, r)$ is an increasing function of r and $rs < r$, we have

$$I_2 \leq C2^{q-1} \int_0^1 (1-r)^{\alpha q-1} M_p^q(T_{b,c} f, r) \left(\int_r^1 (1-s)^{\gamma-1} \mathrm{d}s \right)^q \mathrm{d}r.$$

Then a simple integration for I_2 with respect to s gives

$$I_2 \leq C2^{q-1} \int_0^1 (1-r)^{q(\alpha+\gamma)-1} M_p^q(T_{b,c} f, r) \mathrm{d}r < \infty.$$

Together the above inequalities for I_1 and I_2, and (3) yield the required result. □

If we take $b = 1 + \beta$ and $c = 1$ we get Theorem A of Blasco [3].

Lemma 2.4 *Suppose $b, c \in \mathbb{R}$ with $c > b \geq 1$. Then we have the following inequalities*

(a) *If $p \geq 1$, then*

$$M_p(T_{b,c} f, r) \leq \frac{1}{B(b, c-b)} r^{1-c} \int_0^r s^{b-1} (r-s)^{c-b-1} M_p(f, s) \mathrm{d}s.$$

(b) *If $0 < p < 1$ and $f \in H^p$, then*

$$M_p(T_{b,c} f, r) \leq \frac{1}{B(b, c-b)} \left\{ \int_0^1 (1-s)^{(c-b)p-1} M_p^p(f, rs) \mathrm{d}s \right\}^{\frac{1}{p}}.$$

Proof (a) For $p \geq 1$, from (1) we have

$$|T_{b,c} f(re^{i\theta})| \leq \frac{1}{B(b, c-b)} \int_0^1 s^{b-1} (1-s)^{c-b-1} |f(rse^{i\theta})| \mathrm{d}s.$$

Minkowski's inequality gives the following

$$M_p(T_{b,c}f,r) \le \frac{1}{B(b,c-b)}\frac{1}{2\pi}\int_0^1\left(\int_0^{2\pi} s^{p(b-1)}(1-s)^{p(c-b-1)}|f(rse^{i\theta})|^p\mathrm{d}\theta\right)^{\frac{1}{p}}\mathrm{d}t$$
$$= \frac{1}{B(b,c-b)}\int_0^1 s^{b-1}(1-s)^{c-b-1}M_p(f,rs)\mathrm{d}s$$
$$= \frac{1}{B(b,c-b)}\int_0^r \left(\frac{u}{r}\right)^{b-1}\left(1-\frac{u}{r}\right)^{c-b-1}M_p(f,u)\frac{\mathrm{d}u}{r}$$
$$= \frac{1}{B(b,c-b)}r^{1-c}\int_0^r s^{b-1}(r-s)^{c-b-1}M_p(f,s)\mathrm{d}s.$$

(b) Let $0 < p < 1$. Suppose, $s_k = 1 - 2^{-k}\ \forall\ k \ge 1$. Then $0 = s_0 < s_1 < s_2 < \cdots < 1$ forms a partition of the interval [0,1). We will first show that

$$0 \le \int_{s_{k-1}}^{s_k} s^{b-1}(1-s)^{c-b-1}\mathrm{d}s \le C2^{-k(c-b)}.$$

Here two cases will arise $k = 1$ and $k > 1$. For $k = 1$, since $(1-s)^{c-b-1} = (1-s)^{-1}(1-s)^{c-b} \le 2(1-s)^{c-b} \le 2$, therefore

$$0 \le \int_{s_0}^{s_1} s^{b-1}(1-s)^{c-b-1}\mathrm{d}s < 2\int_0^{\frac{1}{2}} s^{b-1}\mathrm{d}s = \frac{1}{b2^{b-1}}.$$

For $k > 1$, since $s^{b-1} = s^{-1}s^b \le 2s^b \le 2$, therefore

$$0 \le \int_{s_{k-1}}^{s_k} s^{b-1}(1-s)^{c-b-1}\mathrm{d}s < 2\int_{s_{k-1}}^{s_k}(1-s)^{c-b-1}\mathrm{d}s$$
$$= \frac{2}{c-b}[2^{-(k-1)(c-b)} - 2^{-k(c-b)}]$$
$$= \frac{2}{c-b}[2^{(c-b)} - 1]2^{-k(c-b)}$$
$$< 2\frac{2^{c-b}}{c-b}2^{-k(c-b)}.$$

Combining these two cases we get the required inequality. Let $G_k(r,\theta) = \sup_{s\in(s_{k-1},s_k)}|f(rse^{i\theta})|$. Now,

$$|T_{b,c}f(re^{i\theta})| \le \left(\frac{1}{B(b,c-b)}\right)\left|\int_0^1 s^{b-1}(1-s)^{c-b-1}f(rse^{i\theta})\mathrm{d}s\right| \quad \text{(from (1))}$$
$$\le \left(\frac{1}{B(b,c-b)}\right)\sum_{k=1}^{\infty}\int_{s_{k-1}}^{s_k} s^{b-1}(1-s)^{c-b-1}|f(rse^{i\theta})|\mathrm{d}s$$

$$\leq \left(\frac{1}{B(b,c-b)}\right)\sum_{k=1}^{\infty} G_k(r,\theta)\int\limits_{s_{k-1}}^{s_k} s^{b-1}(1-s)^{c-b-1}\mathrm{d}s$$

$$\leq \left(\frac{1}{B(b,c-b)}\right)\sum_{k=1}^{\infty} G_k(r,\theta)2^{-k(c-b)}$$

$$|T_{b,c}f(re^{i\theta})|^p \leq C\left(\frac{1}{B(b,c-b)}\right)^p\sum_{k=1}^{\infty} G_k^p(r,\theta)2^{-k(c-b)p} \quad (\text{since } 0<p<1).$$

Therefore,

$$M_p^p(T_{b,c}f,r) \leq \frac{C}{2\pi}\left(\frac{1}{B(b,c-b)}\right)^p\sum_{k=1}^{\infty} 2^{-k(c-b)p}\int\limits_0^{2\pi} G_k^p(r,\theta)\mathrm{d}\theta$$

$$\leq \frac{C}{2\pi}\left(\frac{1}{B(b,c-b)}\right)^p\sum_{k=1}^{\infty} 2^{-k(c-b)p}\int\limits_0^{2\pi} \sup_{s\in(0,s_k)} |f(rse^{i\theta})|^p\mathrm{d}\theta$$

$$\leq C\left(\frac{1}{B(b,c-b)}\right)^p\sum_{k=1}^{\infty} 2^{-k(c-b)p}M_p^p(f,rs_k) \quad (\text{Lemma 2.1(iii)})$$

$$\leq C\left(\frac{1}{B(b,c-b)}\right)^p\sum_{k=1}^{\infty}\int\limits_{s_k}^{s_{k+1}} (1-s)^{(c-b)p-1}M_p^p(f,rs)\mathrm{d}s$$

$$\leq C\left(\frac{1}{B(b,c-b)}\right)^p\int\limits_0^1 (1-s)^{(c-b)p-1}M_p^p(f,rs)\mathrm{d}s.$$

which completes the proof. □

Theorem 2.5 *For $0<p<\infty$, $0<q,\alpha,\gamma<\infty$, suppose $b,c\in\mathbb{R}$ with $c>b\geq 1$ and $\gamma=c-b$.*
(a) *Let $p\geq 1$. If $f\in H(p,q,\alpha)$, then $T_{b,c}f\in H(p,q,\alpha-\gamma)$.*
(b) *Let $0<p<1$. If $f\in H^p$, then $T_{b,c}f\in H(p,q,\alpha-\gamma)$.*

Proof (a) Let $p\geq 1$ and $f\in H(p,q,\alpha)$. The proof follows from Lemma 2.4(a) in a similar way to case III of Theorem 2.3.
(b) Let $0<p<1$ and $\frac{q}{p}>1$. From Lemma 2.4(b) we have,

$$\|T_{b,c}f\|_{p,q,\alpha-\gamma}^q \leq C\int\limits_0^1 (1-r)^{(\alpha-\gamma)q-1}\left(\int\limits_0^1 (1-s)^{\gamma p-1}M_p^p(f,rs)\mathrm{d}s\right)^{\frac{q}{p}}\mathrm{d}r$$

$$= C\int\limits_0^1 (1-r)^{(\alpha-\gamma)q-1}\left(\int\limits_0^r (1-s)^{\gamma p-1}M_p^p(f,rs)\mathrm{d}s + \int\limits_r^1 (1-s)^{\gamma p-1}M_p^p(f,rs)\mathrm{d}s\right)^{\frac{q}{p}}\mathrm{d}r$$

$$\leq I_1+I_2, \tag{4}$$

where,

$$I_1 = C2^{\frac{q}{p}-1}\int\limits_0^1 (1-r)^{(\alpha-\gamma)q-1}\left(\int\limits_0^r (1-s)^{\gamma p-1}M_p^p(f,rs)\mathrm{d}s\right)^{\frac{q}{p}}\mathrm{d}r$$

and

$$I_2 = C2^{\frac{q}{p}-1} \int_0^1 (1-r)^{(\alpha-\gamma)q-1} \left(\int_r^1 (1-s)^{\gamma p-1} M_p^p(f, rs) \mathrm{d}s \right)^{\frac{q}{p}} \mathrm{d}r.$$

Therefore,

$$I_1 \le C2^{\frac{q}{p}-1} \int_0^1 (1-r)^{(\alpha-\gamma)q-1} \left(\int_0^r (1-s)^{\gamma p-1} M_p^p(f, s) \mathrm{d}s \right)^{\frac{q}{p}} \mathrm{d}r.$$

Taking $k = \frac{q}{p}$, $\mu = p(\alpha - \gamma)$, $\delta = 1$ and $h(s) = (1-s)^{\gamma p-1} M_p^p(f, s)$ in Lemma 2.1(i) gives

$$I_1 \le C2^{\frac{q}{p}-1} \int_0^1 (1-r)^{\alpha q-1} M_p^q(f, r) \mathrm{d}r < \infty.$$

Since integral mean $M_p(f, r)$ of f is an increasing function of r and $rs < r$, we have

$$I_2 \le C2^{\frac{q}{p}-1} \int_0^1 (1-r)^{(\alpha-\gamma)q-1} M_p^q(f, r) \left(\int_r^1 (1-s)^{\gamma p-1} \mathrm{d}s \right)^{\frac{q}{p}} \mathrm{d}r.$$

Then a simple integration for I_2 with respect to s gives

$$I_2 \le C2^{\frac{q}{p}-1} \int_0^1 (1-r)^{\alpha q-1} M_p^q(f, r) \mathrm{d}r < \infty.$$

Together the above inequalities for I_1 and I_2, and (4) yield the required result. Let $0 < p < 1$ and $\frac{q}{p} \le 1$. Similar to the argument as given in the proof of Lemma 2.4(b) together with monotonicity of integral mean $M_p(f, r)$, we have

$$\begin{aligned} M_p^q(T_{b,c} f, r) &\le C \left(\sum_{k=1}^{\infty} \int_{s_{k-1}}^{s_k} (1-s)^{\gamma p-1} M_p^p(f, sr) \mathrm{d}s \right)^{\frac{q}{p}} \\ &\le C \left(\sum_{k=1}^{\infty} M_p^p(f, s_k r) \int_{s_{k-1}}^{s_k} (1-s)^{\gamma p-1} \mathrm{d}s \right)^{\frac{q}{p}} \\ &\le C \int_0^1 (1-s)^{\gamma q-1} M_p^q(f, sr) \mathrm{d}s. \end{aligned}$$

This gives,

$$\begin{aligned} \|T_{b,c} f\|_{p,q,\alpha-\gamma}^q \le C \int_0^1 (1-r)^{(\alpha-\gamma)q-1} &\left(\int_0^r (1-s)^{\gamma q-1} M_p^q(f, sr) \mathrm{d}s \right. \\ &\left. + \int_r^1 (1-s)^{\gamma q-1} M_p^q(f, sr) \mathrm{d}s \right) \mathrm{d}r. \end{aligned}$$

The remaining part of the proof is similar to the case $\frac{q}{p} > 1$, taking $k = 1$, $\mu = q(\alpha - \gamma)$, $\delta = 1$ and $h(s) = (1-s)^{\gamma q-1} M_p^q(f, s)$ in Lemma 2.1(i) for the first integral and the increasing property of integral mean $M_p(f, r)$ for the second integral. □

3 Corollaries

For some special choices of $p, q, \alpha, \beta, a, b, c$ the following results are obtained as corollaries to the above theorems.

Corollary 3.1 *For* $1 \le p < \infty$, $0 < q, \alpha, \gamma < \infty, \alpha \ne 1 - \frac{1}{p}$ *and* $\gamma = a + b - c - 1$, $F(a, b; c; z) \in H(p, q, \alpha + \gamma)$.

Proof Let $f(z) = \frac{1}{1-z}$. Then $M_p(f, r) \le (1-r)^{\frac{1}{p}-1}$. A simple calculation shows that $f(z) \in H(p, q, \alpha)$ for $1 \le p < \infty$, $0 < q, \alpha < \infty$ provided $\alpha \ne 1 - \frac{1}{p}$. By Theorem 2.2,
$f(z) * F(a, b; c; z) \in H(p, q, \alpha + \gamma)$ which gives $F(a, b; c; z) \in H(p, q, \alpha + \gamma)$. □

Corollary 3.2 *Suppose* $0 < q, \alpha < \infty$. *For* $b = 1, c = 1 + \beta$ *in Theorem* 2.5 *we have*

(a) *Let* $p \ge 1$. *If* $f \in H(p, q, \alpha)$, *then* $f_{[\beta]} \in H(p, q, \alpha - \beta)$.
(b) *Let* $0 < p < 1$. *If* $f \in H^p$, *then* $f_{[\beta]} \in H(p, q, \alpha - \beta)$.

Corollary 3.3 [8]. *Let* $0 < p < q < 1$ *and* $\beta = \frac{1}{p} - \frac{1}{q}$.

(a) *If* $f \in B^q$, *then* $f^{[\beta]} = T_{1+\beta,1} f \in B^p$.
(b) *If* $f \in B^p$, *then* $f_{[\beta]} = T_{1,1+\beta} f \in B^q$.

Proof (a) For $p = q = 1, \alpha = \frac{1}{q} - 1$, in Theorem 2.2, we have if $f \in H\left(1, 1, \frac{1}{q} - 1\right) = B^q$, then $f^{[\beta]} \in H\left(1, 1, \frac{1}{q} - 1 + \frac{1}{p} - \frac{1}{q}\right) = B^p$.

(b) For $p = q = 1, \alpha = \frac{1}{p} - 1$, in Theorem 2.5, we have if $f \in H\left(1, 1, \frac{1}{p} - 1\right) = B^p$, then $f_{[\beta]} \in H\left(1, 1, \frac{1}{p} - 1 - \frac{1}{p} + \frac{1}{q}\right) = B^q$. □

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